Macro prelim solutions - June 2016[1]

*Disclaimer: These are unofficial solutions, they might have errors and be incomplete. Your comments and corrections are welcome.*

## Question 1 (Corbae)

*This solution was written together with Fu Tan.*

(a) The planner's problem is given by:

$$\begin{aligned}
\max_{c_1^0,(c_t^t,c_{t+1}^t,K_{t+1})_{t=1}^{\infty}} \quad & \ln(c_1^0) + \sum_{t=1}^{\infty}\left[\ln(c_t^t) - \frac{\gamma}{2}n_t^2 + \beta \ln c_{t+1}^t\right] \\
\text{s.t.} \quad & c_t^{t-1} + c_t^t + K_{t+1} = F(K_t, n_t) \quad \forall t \geq 1 \\
& c_t^{t-1}, c_t^t, K_{t+1} \geq 0 \quad \forall t \geq 1 \\
& K_1 = \overline{K}_1.
\end{aligned}$$

To solve the steady state allocations to the social planner's problem (SPP), write down the Lagrangian and take first order conditions (F.O.C) with respect to $c_1^0$, $c_t^t$, $c_{t+1}^t$,$n_t$, and $K_{t+1}$ for $t \geq 1$.

$$L = \ln(c_1^0) + \sum_{t=1}^{\infty}\left[\ln(c_t^t) - \frac{\gamma}{2}n_t^2 + \beta \ln c_{t+1}^t\right] + \sum_{t=1}^{\infty}\theta_t\left[F(K_t,n_t) - c_t^{t-1} - c_t^t - K_{t+1}\right]$$

F.O.C

$$\frac{1}{c_1^0} = \theta_1$$

$$\frac{1}{c_t^t} = \theta_t \tag{1}$$

$$\frac{\beta}{c_{t+1}^t} = \theta_{t+1} \tag{2}$$

$$\gamma n_t = \theta_t F_L(K_t, n_t) \tag{3}$$

$$\theta_t = \theta_{t+1} F_K(K_{t+1}, n_{t+1}) \tag{4}$$

(1) and (2) imply for $t \geq 1$

$$\begin{aligned}
& \frac{1}{c_{t+1}^t} = \frac{\beta}{c_{t+1}^t} \\
\implies \quad & c_{t+1}^t = \beta c_{t+1}^{t+1}
\end{aligned}$$

(1) and (4) imply for $t \geq 1$

$$\frac{1}{c_t^t} = \frac{1}{c_{t+1}^{t+1}} F_K(K_{t+1}, n_{t+1})$$

$$\implies \quad F_K(K_{t+1}, n_{t+1}) = \frac{c_{t+1}^{t+1}}{c_t^t} \tag{5}$$

(1) and (3) imply for $t \geq 1$

$$\gamma n_t = \frac{1}{c_t^t} F_L(K_t, n_t)$$

$$\implies \quad F_L(K_t, n_t) = \gamma n_t c_t^t \tag{6}$$

[1] By Anton Babkin. This version: July 30, 2016.

Let $C_t$ be the aggregate consumption, i.e. $C_t = c_t^{t-1} + c_t^t$. Then generation $t$'s consumption when young and old are just constant fraction of aggregate consumption, i.e. $c_t^{t-1} = \frac{\beta}{1+\beta} C_t$ and $c_t^t = \frac{1}{1+\beta} C_t$ in the optimal allocation.

As aggregate consumption is constant over time in steady state, consumption when young or old should be the same for any generations in steady state, i.e. $c_t^{t-1} = \bar{c}^o$ and $c_t^t = \bar{c}^y$, which implies $F_K(K_{ss}^{sp}, n_{ss}^{sp}) = \alpha \left( \frac{K_{ss}^{sp}}{n_{ss}^{sp}} \right)^{\alpha-1} = 1$.

We can solve the optimal steady state aggregate capital and labor by solving the following systems of equations

$$
\begin{aligned}
(1+\beta)\bar{c}^y + K_{ss}^{sp} &= (K_{ss}^{sp})^\alpha (n_{ss}^{sp})^{1-\alpha} \\
\alpha \left( \frac{K_{ss}^{sp}}{n_{ss}^{sp}} \right)^{\alpha-1} &= 1 \\
(1-\alpha) \left( \frac{K_{ss}^{sp}}{n_{ss}^{sp}} \right)^{\alpha} &= \gamma n_{ss}^{sp} \bar{c}^y \\
\implies \quad K_{ss}^{sp} &= \alpha^{\frac{1}{1-\alpha}} \sqrt{\frac{1+\beta}{\gamma}} && (7) \\
\implies \quad n_{ss}^{sp} &= \sqrt{\frac{1+\beta}{\gamma}} && (8)
\end{aligned}
$$

(b) The households' problem for the initial old is trivial.

Generation $t \geq 1$ households' problem are,

$$
\begin{aligned}
\max_{c_t^t \geq 0, c_{t+1}^t \geq 0, s_t \geq 0} \quad & \ln c_t^t - \frac{\gamma}{2} n_t^2 + \beta \ln c_{t+1}^t \\
\text{s.t.} \quad & c_t^t + s_t = w_t(1-\tau_t) n_t + T_t && (9) \\
& c_{t+1}^t = R_{t+1} s_t + \pi_{t+1} && (10)
\end{aligned}
$$

To solve generation $t \geq 1$ households' problem, write down the Lagrangian with $\lambda_t$ and $\mu_t$ as multipliers on (9) and (10) and take F.O.Cs with respect to $c_t^t$, $c_{t+1}^t$, $n_t$, and $k_{t+1}^t$[2]

$$
L = \ln c_t^t - \frac{\gamma}{2} n_t^2 + \beta \ln c_{t+1}^t + \lambda_t [w_t(1-\tau_t) n_t + T_t - c_t^t + s_t] + \mu_t [R_{t+1} s_t + \pi_{t+1} - c_{t+1}^t]
$$

F.O.C

$$
\begin{aligned}
\frac{1}{c_t^t} &= \lambda_t && (11) \\
\frac{\beta}{c_{t+1}^t} &= \mu_t && (12) \\
\gamma n_t &= \lambda_t (1-\tau_t) w_t && (13) \\
\lambda_t &= R_{t+1} \mu_t && (14)
\end{aligned}
$$

Plugging (11) and (12) into (14) gives you the inter-temporal Euler equation,

$$
\frac{c_{t+1}^t}{c_t^t} = R_{t+1} \beta \tag{15}
$$

[2] We first omit the multipliers on the boundary conditions on $k_{t+1}^t$ and then check whether households may choose corner solution of capital holding in the equilibrium.

Plugging (11) into (13) gives you the intra-temporal Euler equation,

$$\gamma c_t^t n_t = w_t(1-\tau_t) \tag{16}$$

To solve for the labor supply policy function, we first consolidate (9) and (10) to obtain the life-time budget constraint[3],

$$R_{t+1}c_t^t + c_{t+1}^t = R_{t+1}w_t n_t + R_{t+1}(T_t - \tau_t w_t n_t) + \pi_{t+1} \tag{17}$$

Then by using the Euler equations (15) and the life-time budget constraint (17), we can solve consumption when young:

$$c_t^t(w_t, n_t, T_t, \tau_t, \pi_{t+1}) = \frac{w_t n_t}{1+\beta} + \frac{T_t - \tau_t w_t n_t}{1+\beta} + \frac{\pi_{t+1}}{(1+\beta)R_{t+1}} \tag{18}$$

Using (16) and (18), we can solve the optimal labor supply:

$$n_t = \frac{-\left(T_t + \frac{\pi_{t+1}}{R_{t+1}}\right) + \sqrt{\left(T_t + \frac{\pi_{t+1}}{R_{t+1}}\right)^2 + 4(1-\tau_t)^2 w_t^2 \frac{(1+\beta)}{\gamma}}}{2(1-\tau_t)w_t} \tag{19}$$

$$= -\frac{T_t + \frac{\pi_{t+1}}{R_{t+1}}}{2(1-\tau_t)w_t} + \sqrt{\left[\frac{T_t + \frac{\pi_{t+1}}{R_{t+1}}}{2(1-\tau_t)w_t}\right]^2 + \frac{(1+\beta)}{\gamma}} \tag{20}$$

From now on, we assume profits are zero. The optimal decision rule for labor supply becomes:

$$n_t = -\frac{T_t}{2(1-\tau_t)w_t} + \sqrt{\left[\frac{T_t}{2(1-\tau_t)w_t}\right]^2 + \frac{(1+\beta)}{\gamma}} \tag{21}$$

We take partial derivative of the optimal labor supply with respect to wages:

$$\frac{\partial n_t}{\partial w_t} = \frac{T_t}{2(1-\tau_t)w_t^2\sqrt{\left[\frac{T_t}{2(1-\tau_t)w_t}\right]^2 + \frac{(1+\beta)}{\gamma}}}\left[\sqrt{\left[\frac{T_t}{2(1-\tau_t)w_t}\right]^2 + \frac{(1+\beta)}{\gamma}} - \frac{T_t}{2(1-\tau_t)w_t}\right] > 0 \tag{22}$$

As the optimal labor supply increases in wages, the labor supply policy function is upward sloping in wages.

(c) Firms' problem is to maximize profits each period.

$$\max_{K_t, L_t} \Pi_t = F(K_t, L_t) - w_t L_t - R_t K_t$$

Firms' first order conditions with respect to labor and capital yield labor demand and capital demand

$$w_t = F_L(K_t^d, L_t^d) = (1-\alpha)(K_t^d)^\alpha (L_t^d)^{-\alpha} \tag{23}$$
$$R_t = F_K(K_t^d, L_t^d) = \alpha(K_t^d)^{\alpha-1}(L_t^d)^{1-\alpha} \tag{24}$$

With constant return to scale in the production function, firms earn zero profits at the optimal, i.e. $\Pi_t = F(K_t^d, L_t^d) - w_t L_t^d - R_t K_t^d = 0$.

[3]Note that young households take transfers and rebate given. In households' problem, we do not take market clearing conditions into account.

(d) Market clearing conditions for the final consumption good, capital, and labor are for all $t$,

$$\begin{aligned}
\text{Demand} &= \text{Supply} \\
c_t^t + c_t^{t-1} + s_t &= F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha} && (25) \\
K_t &= s_{t-1} && (26) \\
L_t &= n_t && (27)
\end{aligned}$$

An equilibrium is a sequence
$\{c_1^{0*}, s_1, (c_t^{t*}, n_t^*, c_{t+1}^{t*}, s_t^*)_{t=1}^{\infty}, (C_t^*, K_t^*, L_t^*, Y_t^*, \Pi_t^*)_{t=1}^{\infty}, (r_t^*, w_t^*)_{t=1}^{\infty}\}$ such that:
(1) Initial old optimize: $c_1^{0*}$ solves

$$\begin{aligned}
&\max_{c_1^0} \ln c_1^0 \\
\text{s.t. } & c_1^0 = r_1^* s_0 + \Pi_1^* \\
& s_0 = \bar{K}_1 \\
& c_1^0 \geq 0
\end{aligned}$$

(2) All subsequent households optimize: given $(r_{t+1}^*, \Pi_{t+1}^*, \tau_t, T_t)$, $(c_t^{t*}, c_{t+1}^{t*}, n_t^*, s_t^*)$ for all $t \geq 1$solves

$$\begin{aligned}
&\max_{(c_t^t, c_{t+1}^t, n_t, s_t)} \ln c_t^t + \beta ln c_{t+1}^t \\
\text{s.t. } & c_t^t + s_t + k_{t+1}^t \leq (1-\tau_t) w_t^* n_t + T_t \\
& c_{t+1}^t \leq b_{t+1}^t + r_{t+1}^* s_t + \Pi_{t+1}^* \\
& c_t^t, c_{t+1}^t, n_t, 1 - n_t \geq 0 \\
& s_t \geq 0
\end{aligned}$$

(3) The firm maximizes profits: given $(w_t^*, r_t^*)$, $K_t^*$ and $L_t^*$ for all $t \geq 1$ solve

$$\Pi_t^* = \max_{K_t, L_t \geq 0} K_t^{\alpha} L_t^{1-\alpha} - w_t^* L_t - r_t^* K_t$$

(4) Markets clear: for all $t$,

$$\begin{aligned}
c_t^{t*} + c_t^{t-1*} + s_t^* &= F(K_t^*, L_t^*) \\
s_{t-1}^* &= K_t^* \\
n_t^* &= L_t^*
\end{aligned}$$

(5) Government budget constraint holds: $T_t = \tau_t w_t n_t$.

(e) Using the Euler equations (15) and the expression for consumption when young (18), we can solve consumption when old: when old,

$$c_{t+1}^t(w_t, r_{t+1}) = \frac{\beta r_{t+1} w_t n_t}{1+\beta}. \tag{28}$$

By the second-period budget constraint (10), the optimal saving is a function of wage and labor supply when young

$$s_t(w_t) = \frac{\beta w_t n_t}{1+\beta}. \tag{29}$$

Using (27), (21) and the government budget constraint, we can solve the equilibrium labor:

$$L_t = n_t = \sqrt{\frac{(1-\tau_t)(1+\beta)}{\gamma}} \tag{30}$$

The market clearing condition for capital (26) at time $t+1$ and the optimal decision rule for saving (29) give

$$\begin{aligned} K_{t+1} &= s_t \\ &= \frac{\beta}{1+\beta} w_t n_t \\ &= \frac{\beta}{1+\beta}(1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha} L_t \\ \Longrightarrow K_{t+1} &= \frac{(1-\alpha)\beta}{1+\beta} K_t^{\alpha}\left[\frac{(1-\tau_t)(1+\beta)}{\gamma}\right]^{\frac{1-\alpha}{2}} \end{aligned} \tag{31, 32}$$

where the third line is derived by substituting $w_t$ and $L_t$ with (23) and (30).

The law of motion for capital is shown in Figure e. The $K_{t+1}(K_t)$ curve cross the 45-degree line twice.

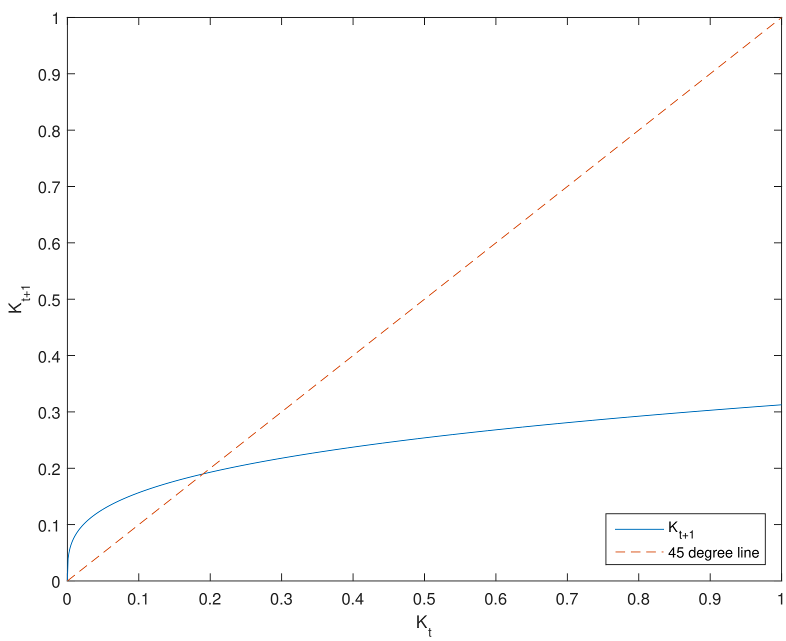


Figure 1: Law of motion for capital

There are two steady states:

$$K_1^{ss} = 0 \tag{33}$$

$$K_2^{ss} = \left[\frac{(1-\tau_t)(1+\beta)}{\gamma}\right]^{\frac{1}{2}}\left[\frac{(1-\alpha)\beta}{1+\beta}\right]^{\frac{1}{1-\alpha}} \tag{34}$$

(f) Allocations in the steady state competitive equilibrium are not the same as those in the planner's problem when $\alpha \leq \frac{\beta}{1+2\beta}$. The government can set the labor income tax $\tau = 1 - \left[\frac{\alpha(1+\beta)}{\beta(1-\alpha)}\right]^{\frac{2}{1-\alpha}}$ to implement the planner's solution.

## Question 2 (Seshadri)

1. Bellman equation:

$$V(k,A) = \max_{c,k',l} \ln c + b\ln(1-l) + \beta \mathbb{E} V(k',A')$$
$$\text{s.t. } c + k' = Ak^\alpha l^{1-\alpha}$$

2.

$$\text{FOC } [l]: \frac{(1-\alpha)Ak^\alpha l^{-\alpha}}{c} = \frac{b}{1-l} \tag{35}$$
$$\text{FOC } [k']: \frac{1}{c} = \beta \mathbb{E} V_1(k',A') \tag{36}$$
$$\text{ENV } [k]: V_1(k,A) = \frac{\alpha A k^{\alpha-1} l^{1-\alpha}}{c} \tag{37}$$

Combine (36) and (37) for Euler equation:

$$\frac{1}{c} = \mathbb{E}\beta \frac{1}{c'} \alpha A k'^{\alpha-1} l'^{1-\alpha}$$

3. Guess value function to be $V(k,A) = D + E\ln k + F\ln A$, and capital policy function to be $k' = GAk^\alpha l^{1-\alpha}$, i.e. save costant fraction of output. No guess for labor policy function, we are going to derive it. *Alternatively, make a guess that labor supply is a constant.*

ENV $[k]$:

$$\frac{E}{k} = \frac{\alpha A k^{\alpha-1} l^{1-\alpha}}{(1-G)Ak^\alpha l^{1-\alpha}}$$

Solve for $G = \alpha\beta$.

FOC $[l]$ becomes:

$$\frac{(1-\alpha)Ak^\alpha l^{-\alpha}}{(1-G)Ak^\alpha l^{1-\alpha}} = \frac{b}{1-l}$$

This can be solved for $l = 1 - \frac{b(1-G)}{b(1-G)+1-\alpha} = 1 - \frac{b(1-\alpha\beta)}{b(1-\alpha\beta)+1-\alpha}$, so it is a constant. Let's call it $L$.

FOC $[k']$:

$$\frac{1}{(1-G)Ak^\alpha l^{1-\alpha}} = \beta \frac{E}{GAk^\alpha l^{1-\alpha}}$$

This simplifies to $E = \frac{G}{\beta(1-G)} = \frac{\alpha}{1-\alpha\beta}$.

Now turn back to the Bellman equation:

$$D + E\ln k + F\ln A = \max_{c,k',l} \ln c + b\ln(1-l) + \beta\mathbb{E}(D + E\ln k' + F\ln A')$$

Under our guess, optimal policy $l = L$ and $k' = GAk^\alpha l^{1-\alpha}$. Also remember that expectation here is conditional on $A$, so $\mathbb{E}\ln A' = \mathbb{E}[\rho \ln A + \epsilon] = \rho\ln A$. Bellman equation becomes

$$D + E\ln k + F\ln A = \ln((1-G)Ak^\alpha L^{1-\alpha}) + b\ln(1-L) + \beta D + \beta E\ln(GAk^\alpha L^{1-\alpha}) + F\rho\ln A$$

Group terms together to apply method of undetermined coefficients:

$$D+E\ln k+F\ln A = \ln((1-G)L^{1-\alpha}) + b\ln(1-L) + \beta D + \beta E\ln(GL^{1-\alpha}) + (\alpha+\alpha\beta E)\ln k + (1+\beta E+F\rho)\ln A$$

Now we can write three equations that can be solved for remaining unknown coefficients $E, F$ and $D$. (We actually already found $E$, so this is a double-check).

$$
\begin{aligned}
D &= \ln((1-G)L^{1-\alpha}) + b\ln(1-L) + \beta D + \beta E \ln(GL^{1-\alpha}) \\
E &= \alpha + \alpha\beta E \\
F &= 1 + \beta E + F\rho
\end{aligned}
$$

It is easy to solve for $E = \frac{\alpha}{1-\alpha\beta}$ and $F = \frac{1}{(1-\rho)(1-\alpha\beta)}$. *D is a mess, and I did not finish it: it is not interesting as long as it does not depend on ρ.*

To conclude, the value function and policy functions are

$$
\begin{aligned}
V(k, A) &= D + \frac{\alpha}{1-\alpha\beta}\ln k + \frac{1}{(1-\rho)(1-\alpha\beta)}\ln A \\
l = L &= 1 - \frac{b(1-\alpha\beta)}{b(1-\alpha\beta)+1-\alpha} \\
k' &= \alpha\beta A k^{\alpha} L^{1-\alpha} \\
c &= (1-\alpha\beta)Ak^{\alpha}L^{1-\alpha}
\end{aligned}
$$

4. Turns out that $\rho$ only positively affects sensitivity of the value function to $A$ and does not affect policy functions.

   Value function is a present discounted value of all future utilities. When there is a positive $A$ shock, production, consumption and savings will all go up and current utility will go up. If $\rho$ is high, positive shock will persist into the future, positively affecting consumption and utility in future periods, so effect on the value function will be high. But if the positive shock decays quickly (low $\rho$), positive effect of $A$ on $V$ will be weaker. Analogously one can exaplain a negative shock.

   *And I don't have good intuition for why k and l are independent of rho.*

## Question 3 (Seshadri)

(a) Endogenous state variables: $k_t, s_{t-1}$, exogenous state $A_t$, choice $c_t, s_t, l_t$.

Bellman equation:

$$
\begin{aligned}
V(k_t, s_{t-1}, A_t) &= \max_{c_t, s_t, l_t} u(c_t) + \beta V(k_{t+1}, s_t, A_{t+1}) \\
&\text{s.t. } c_t + s_t = A_t F(k_t, l_t) \\
&k_{t+1} = (1-\delta)k_t + s_{t-1}
\end{aligned}
$$

Nothing is said in the problem about constraints on labor supply $l_t$. Since it does not enter utility function, optimal allocation of labor is in on the upper bound of feasibility set. I will not consider it in the rest of the problem.

Technology parameter $A_t$ does not play any role either, but it's included to accomodate any deterministic process.

(b) Rewrite Bellman equation with constraints plugged in.

$$
V(k_t, s_{t-1}, A_t) = \max_{s_t} u(A_t F(k_t, l_t) - s_t) + \beta V((1-\delta)k_t + s_{t-1}, s_t, A_{t+1})
$$

$$
\begin{aligned}
\text{FOC } [s_t] &: u'(c_t) = \beta V_2(k_{t+1}, s_t, A_{t+1}) && (38) \\
\text{ENV } [s_{t-1}] &: V_2(k_t, s_{t-1}, A_t) = \beta V_1(k_{t+1}, s_t, A_{t+1}) && (39) \\
\text{ENV } [k_t] &: V_1(k_t, s_{t-1}, A_t) = u'(c_t)A_t F_1(k_t, l_t) + \beta(1-\delta)V_1(k_{t+1}, s_t, A_{t+1}) && (40)
\end{aligned}
$$

(c) Substitute $V_2$ in (38) from (39):

$$u'(c_t) = \beta^2 V_1(k_{t+2}, s_{t+1}, A_{t+2})$$

Update (40) two periods forward and multiply by $\beta^2$:

$$\beta^2 V_1(k_{t+1}, s_{t+1}, A_{t+2}) = \beta^2 u'(c_{t+1}) A_{t+2} F_1(k_{t+2}, l_{t+2}) + \beta^3(1-\delta) V_1(k_{t+3}, s_{t+2}, A_{t+3})$$

Combine two equations into an Euler equation:

$$u'(c_t) = \beta(1-\delta)u'(c_{t+1}) + \beta^2 u'(c_{t+2}) A_{t+2} F_1(k_{t+2}, l_{t+2})$$

The RHS has two terms. The second term has standard interpretation: marginal utility of giving up consumption now should be equal to the present value of marginal utility of having more capital, and hence output and consumption, two periods from now.

Now for the first term. As we save a little more in $t$, we'll have a bit more capital in $t+2$. This in turn means that we'll have a bit more times $(1-\delta)$ capital in $t+3$. This means, we can save that much less in $t+1$ and enjoy higher consumtion.

# Question 4 (Williams)

(a) Price is a function of exogenous aggregate states $X$ and $S$, $p(S_t, X_t)$.

Recursive problem:

$$\begin{aligned} V(a_t, S_t, X_t) = \max_{C_t, a_{t+1}} U(C_t, X_t) + \beta \mathbb{E}_{S_{t+1}}[V(a_{t+1}, S_{t+1}, X_{t+1}) | S_t] \\ \text{s.t. } C_t + p_t a_{t+1} = (p_t + S_t) a_t \\ X_{t+1} = F(X_t, S_t, S_{t+1}) \end{aligned}$$

We want to solve an original sequential problem, but under certain standard assumptions the Principle of optimality holds which guarantees that solution of the recursive problem also solves sequential problem. We would usually also allow additional stronger assumptions under which solution to the recursive problem is unique, and Principle of optimality holds too.

*We did not spend much time on stochastic dynamic programming in class. The theory is presented, for example, in Chapter 9 of Stokey, Lucas, Prescott.*

Plug in constraints:

$$V(a_t, S_t, X_t) = \max_{a_{t+1}} U((p_t + S_t)a_t - p_t a_{t+1}, X_t) + \beta \mathbb{E}_{S_{t+1}}[V(a_{t+1}, S_{t+1}, F(X_t, S_t, S_{t+1})) | S_t]$$

Deriving optimality conditions:

$$\begin{aligned} \text{FOC}[a_{t+1}] : p_t U_1(C_t, X_t) = \beta \mathbb{E}_{S_{t+1}}[V_1(a_{t+1}, S_{t+1}, X_{t+1}) | S_t] \\ \text{ENV}[a_t] : V_1(a_t, S_t, X_t) = (p_t + S_t) U_1(C_t, X_t) \end{aligned}$$

Combine for a standard asset pricing Euler equation:

$$p_t U_1(C_t, X_t) = \beta \mathbb{E}_{S_{t+1}}[(p_{t+1} + S_{t+1}) U_1(C_{t+1}, X_{t+1}) | S_t]$$

(b) Recursive competitive equilibrium is a set of value function $V(a, S, X)$, policy functions $a'(a, S, X)$, $C(a, S, X)$ and price function $p(S, X)$, such that

- $V$, $a'$ and $C$ solve recursive problem of the representative agent taking price funcion $p$ as given,
- Markets clear: $a = 1$, $C = S$.

(c) Rearrange the Euler equation and use goods market clearing:

$$p_t = \beta \mathbb{E}_{S_{t+1}}[\frac{U_1(S_{t+1}, X_{t+1})}{U_1(S_t, X_t)}(p_{t+1} + S_{t+1})|S_t]$$

Iterating this equation forward and rulling out bubble solutions, can obtain a standard asset pricing equation:

$$p_t = \mathbb{E}[\sum_{i=1}^{\infty} \beta^i \frac{U_1(S_{t+i}, X_{t+i})}{U_1(S_t, X_t)} S_{t+i}|S_t]$$

where $m_{t+i} \equiv \beta^i \frac{U_1(S_{t+i}, X_{t+i})}{U_1(S_t, X_t)}$ is known as the stochastic discount factor. This equation states that the current price of the asset must be equal to the sum of all future dividend flows from that asset, appropriatelly discounted.

Once we have the asset pricing formula, we can use it to price any general claim, including a risk-free bond:

$$q_t = \beta \mathbb{E}_{S_{t+1}}[\frac{U_1(S_{t+1}, X_{t+1})}{U_1(S_t, X_t)}|S_t]$$

Risk-free gross interest rate is $R_t = 1/q_t$. It will be higher if dividends are expected to rise, and lower otherwise.

(d) Now $U_1(C, X) = C^{-\gamma}$.

$$\begin{aligned} q_t &= \beta \mathbb{E}\left[\left(\frac{S_{t+1}}{S_t}\right)^{-\gamma}\right] \\ &= \beta \mathbb{E}\left[\left(e^{s_{t+1}-s_t}\right)^{-\gamma}\right] \\ &= \beta \mathbb{E}[e^{-\gamma g - \gamma v_{t+1}}] \\ &= \beta e^{-\gamma g + \frac{\gamma^2 \sigma^2}{2}} \end{aligned}$$

where the last equality uses expectation of lognormal random variable.

$$R = \beta^{-1} e^{\gamma g - \frac{\gamma^2 \sigma^2}{2}}$$

(e) Expectation of a product of dependent random variables:

$$\mathbb{E}(m_{t+1}, R^e_{t+1}) = \mathbb{E}(m_{t+1})\mathbb{E}(R^e_{t+1}) + Cov(m_{t+1}, R^e_{t+1})$$

Definition of correlation coefficient:

$$corr(m_{t+1}, R^e_{t+1}) = \frac{Cov(m_{t+1}, R^e_{t+1})}{\sigma(m_{t+1})\sigma(R^e_{t+1})}$$

Using $\mathbb{E}(m_{t+1}, R^e_{t+1}) = 0$,

$$\mathbb{E}(m_{t+1})\mathbb{E}(R^e_{t+1}) = -corr(m_{t+1}, R^e_{t+1})\sigma(m_{t+1})\sigma(R^e_{t+1})$$

Then the Sharpe ratio is:

$$\frac{\mathbb{E}(R^e_{t+1})}{\sigma(R^e_{t+1})} = -corr(m_{t+1}, R^e_{t+1})\frac{\sigma(m_{t+1})}{\mathbb{E}(m_{t+1})}$$

Since correlation coefficient is bounded in $[-1, 1]$, the maximal Sharpe ratio equals $\frac{\sigma(m_{t+1})}{\mathbb{E}(m_{t+1})}$.

(f) We derived in part (d) that $\mathbb{E}(m_{t+1}) = \beta e^{-\gamma g + \frac{\gamma^2\sigma^2}{2}}$.

Application of a special case of the expectation of a product formula yields:

$$\begin{aligned}
Var[m_{t+1}] &= \mathbb{E}[m_{t+1}^2] - (\mathbb{E}[m_{t+1}])^2 \\
&= \mathbb{E}[(\beta e^{-\gamma g - \gamma v_{t+1}})^2] - (\beta e^{-\gamma g + \frac{\gamma^2\sigma^2}{2}})^2 \\
&= \beta^2 \mathbb{E}[e^{-2\gamma g - 2\gamma v_{t+1}}] - \beta^2 e^{-2\gamma g + \gamma^2\sigma^2} \\
&= \beta^2 e^{-2\gamma g + 2\gamma^2\sigma^2} - \beta^2 e^{-2\gamma g + \gamma^2\sigma^2} \\
&= \beta^2 e^{-2\gamma g + \gamma^2\sigma^2}(e^{\gamma^2\sigma^2} - 1)
\end{aligned}$$

$$\sigma(m_{t+1}) = \beta e^{-\gamma g + \frac{\gamma^2\sigma^2}{2}}(e^{\gamma^2\sigma^2} - 1)^{1/2}$$

So the maximal Sharpe ratio is $(e^{\gamma^2\sigma^2} - 1)^{1/2}$. This expression demonstrates that higher return on risky asset (like equity), normalized by standard deviation, requires higher risk aversion, parametrized by $\gamma$. Empirical estimates of similar relationships conclude that observed stock returns would be only justified by extremely high risk aversion (high $\gamma$), much higher than predicted by micro-level studies. This constitutes the equity premium puzzle.

(g) Now $U_1(C, X) = C^{-\gamma} X^{1-\gamma}$.

$$\begin{aligned}
m_{t+1} &= \beta \frac{S_{t+1}^{-\gamma} X_{t+1}^{1-\gamma}}{S_t^{-\gamma} X_t^{1-\gamma}} \\
&= \beta(\exp(s_{t+1} - s_t))^{-\gamma}(\exp(x_{t+1} - x_t))^{1-\gamma} \\
&= \beta \exp(-\gamma g - \gamma v_{t+1}) \exp((1-\gamma)(1-\phi)(\bar{x} - x_t) + (1-\gamma)\lambda(x_t) v_{t+1}) \\
&= \beta \exp((1-\gamma)(1-\phi)(\bar{x} - x_t) - \gamma g + ((1-\gamma)\lambda(x_t) - \gamma) v_{t+1})
\end{aligned}$$

So $m_{t+1} \sim lnN((1-\gamma)(1-\phi)(\bar{x} - x_t) - \gamma g, ((1-\gamma)\lambda(x_t) - \gamma)^2\sigma^2)$, conditional on $X_t$.

Then

$$\mathbb{E}[m_{t+1}] = \beta \exp((1-\gamma)(1-\phi)(\bar{x} - x_t) - \gamma g + \frac{((1-\gamma)\lambda(x_t) - \gamma)^2\sigma^2}{2})$$

The gross risk-free rate is $R_t = 1/\mathbb{E}[m_{t+1}]$.

And

$$\mathbb{E}[m_{t+1}^2] = \beta^2 \exp(2(1-\gamma)(1-\phi)(\bar{x} - x_t) - 2\gamma g + 2((1-\gamma)\lambda(x_t) - \gamma)^2\sigma^2)$$

So using the same formula for covariance,

$$Var[m_{t+1}] = \beta^2 \exp\left(2(1-\gamma)(1-\phi)(\bar{x} - x_t) - 2\gamma g + ((1-\gamma)\lambda(x_t) - \gamma)^2\sigma^2\right)\left(\exp(((1-\gamma)\lambda(x_t) - \gamma)^2\sigma^2) - 1\right)$$

Finally, the maximal Sharpe ratio is now $\left(\exp(((1-\gamma)\lambda(x_t) - \gamma)^2\sigma^2) - 1\right)^{1/2}$. Comparing with the result of part (e), we can see that if levels of function $\lambda(x)$ are sufficiently high, we don't need $\gamma$ to be as high as in (e) to match observed levels of the Sharpe ratio. Intuitively it means that preference parameter $X$ should be sufficiently correlated with the dividend $S$, i.e. agents utility is exogenouosly higher when economy is booming.

Since we don't need high levels of $\gamma$ anymore, both equity premium and risk-free rate puzzle can be resolved. Risk-aversion can be more in line with evidence from micro studies (equity premium puzzle). And marginal rate of intertemporal substitution (another role played by $\gamma$ in the CRRA utility) can be lower, so risk-free rates don't need to be too high (risk-free rate puzzle).

## Question 5 (Atalay)

(a)

$$\begin{aligned}\hat{V}(A,k,\phi) = \max_{k'} \frac{R-1}{\alpha R} A^{1-\alpha}(k')^{\alpha} - (k'-k) - \phi \cdot f \cdot k \cdot 1_{k\neq k'} \\ + R^{-1}\mathbb{E}\left[\lambda \hat{V}(A',k',0) + (1-\lambda)\hat{V}(A',k',1)\right]\end{aligned}$$

(b)

$$\begin{aligned}\hat{V}(A,Az,\phi) = \max_{z'} \frac{R-1}{\alpha R} A(z')^{\alpha} - A\cdot(z'-z) - \phi \cdot f \cdot A \cdot z \cdot 1_{z\neq z'} \\ + R^{-1}\mathbb{E}\left[\lambda \hat{V}(A',Az',0) + (1-\lambda)\hat{V}(A',Az',1)\right]\end{aligned}$$

So

$$\begin{aligned}\frac{V(A,z,\phi)}{A} &= \max_{z'} \frac{R-1}{\alpha R}(z')^{\alpha} - (z'-z) - \phi \cdot f \cdot z \cdot 1_{z\neq z'} \\ &+ R^{-1}\mathbb{E}\left[\frac{1}{A}\lambda V\left(A', \frac{A}{A'}z', 0\right) + \frac{1}{A}(1-\lambda)V\left(A, \frac{A}{A'}z', 1\right)\right] \\ &= \max_{z'} \frac{R-1}{\alpha R}(z')^{\alpha} - (z'-z) - \phi \cdot f \cdot z \cdot 1_{z\neq z'} \\ &+ R^{-1}\mathbb{E}\left[\frac{A'}{A}\frac{V\left(A', \frac{A}{A'}z', 0\right)}{A'}\lambda + \frac{A'}{A}\frac{V\left(A, \frac{A}{A'}z', 1\right)}{A'}(1-\lambda)\right]\end{aligned}$$

(c)

$$\begin{aligned}v(z,\phi) = \max_{z'} \frac{R-1}{\alpha R}(z')^{\alpha} - (z'-z) - \phi \cdot f \cdot z \cdot 1_{z\neq z'} \\ + R^{-1}\lambda\gamma\mathbb{E}\left[v\left(\frac{z'}{\gamma},0\right)\right] + R^{-1}(1-\lambda)\gamma\mathbb{E}\left[v\left(\frac{z'}{\gamma},1\right)\right]\end{aligned}$$

(d)

$$\tilde{z}^{\alpha-1} = \frac{R}{R-1} - \frac{\lambda}{R-1}\mathbb{E}\left[v'\left(\frac{\tilde{z}}{\gamma},0\right)\right] - \frac{1-\lambda}{R-1}\mathbb{E}\left[v'\left(\frac{\tilde{z}}{\gamma},1\right)\right]$$

(e) If the firm decides to adjust when $\phi = 1$:

$$\begin{aligned}v(z,1) = (1-f)\cdot z + \\ \max_{z'}\left\{\frac{R-1}{\alpha R}(z')^{\alpha} - z' + R^{-1}\lambda\gamma\mathbb{E}\left[v\left(\frac{z'}{\gamma},0\right)\right] + R^{-1}(1-\lambda)\gamma\mathbb{E}\left[v\left(\frac{z'}{\gamma},1\right)\right]\right\}\end{aligned}$$

Computing the FOC also gives

$$(z^*)^{\alpha-1} = \frac{R}{R-1} - \frac{\lambda}{R-1}\mathbb{E}\left[v'\left(\frac{z^*}{\gamma},0\right)\right] - \frac{1-\lambda}{R-1}\mathbb{E}\left[v'\left(\frac{z^*}{\gamma},1\right)\right]$$

So $z^*$ and $\tilde{z}$ are the same.

(f) Since $\lambda = 1$, we only need to consider the $\phi = 0$ event:

$$v(z,0) = \max_{z'} \frac{R-1}{\alpha R}(z')^{\alpha} - (z'-z) + R^{-1}\lambda\gamma\mathbb{E}\left[v\left(\frac{z'}{\gamma},0\right)\right]$$

From part (d)

$$\tilde{z}^{\alpha-1} = \frac{R}{R-1} - \frac{1}{R-1}\mathbb{E}\left[v'\left(\frac{\tilde{z}}{\gamma}, 0\right)\right]$$

Also compute the envelope condition

$$v'(z, 0) = 1$$

Plug this into the FOC

$$\tilde{z} = 1$$

(g) With $\lambda = 0$, firm only invests when outside of inaction region. The investment rates could be either zero or take two values $z^* - \bar{z}$ and $z^* - \underline{z}$. The distribution of investment rates will look bimodal with a spike of mass at 0.

With $\lambda \in (0, 1)$, there are again two peaks in the distribution of investment rates at $z^* - \bar{z}$ and $z^* - \underline{z}$ (the values of $\underline{z}$ and $\bar{z}$ will vary with $\lambda$ though). In addition, there will be a smaller mass of investment rates around 0; these latter investment rates represent the firms that drew $\phi = 0$ and are able to invest without paying the fixed cost.

As $\lambda \to 1$, distribution of investment will mirror the distribution of productivity shocks.