Macro prelim solutions - June 2012[1]

Disclaimer: These are unofficial solutions, they might have errors and be incomplete. Your comments and corrections are welcome.

Question 2A

Looks like there is a typo in the utility function, and I change it to the following:

$$u(t, c_t) = \left(\Pi_{s=1}^t \beta_s\right) \log(c_t)$$

I also assume that in the recursive problem, discount factor for the next period is unknown when decision is made, $V(\beta) = \max u + \mathbb{E}\beta' V(\beta')$. Another, possibly even better, interpretation could be $V(\beta) = \max u + \beta \mathbb{E} V(\beta')$.

(a) Recursive problem:

$$V(d_t, \beta_t) = \max \log(c_t) + \mathbb{E}\beta_{t+1} V(d_{t+1}, \beta_{t+1})$$
$$\text{s.t. } c_t + P_t a_{t+1} + q_t b_{t+1} = (P_t + d_t) a_t + b_t$$

Standard asset pricing Euler equations with market clearing condition $c_t = d_t$:

$$P_t = \mathbb{E}_t \beta_{t+1} \frac{d_t}{d_{t+1}} (P_{t+1} + d_{t+1})$$
$$q_t = \mathbb{E}_t \beta_{t+1} \frac{d_t}{d_{t+1}}$$

With β and d following Markov process, state at t has information about state at $t+1$. For example, if process is persistent and β_t is high, then $\mathbb{E}_t \beta_{t+1}$ is also high. Then then agents are more patient, and prices of both assets will be higher, risk-free rate $R = 1/q$ and expected return on the tree - lower. When d_t is high, it will go down in expectation, so $\mathbb{E}_t \frac{d_t}{d_{t+1}}$ is higher, so the prices are higher.

(b)

$$q_t = \mathbb{E}_t \beta_{t+1} \frac{d_t}{d_{t+1}}$$
$$= d_t \left(\pi \frac{\beta_1}{d_1} + (1-\pi) \frac{\beta_2}{p_2} \right)$$

Risk-free rate:

$$R_t = \frac{1}{q_t} = \frac{1}{d_t \left(\pi \frac{\beta_1}{d_1} + (1-\pi) \frac{\beta_2}{p_2} \right)}$$

[1] By Anton Babkin. This version: June 11, 2016.

(c) Risk-free rate with constant β will be lower if

$$\beta(\frac{\pi}{d_1} + \frac{1-\pi}{d_2}) > \pi\frac{\beta_1}{d_1} + (1-\pi)\frac{\beta_2}{p_2}$$

Simplifying this inequality yields

$$\beta_2 > \beta_1$$

Question 3

Parts of this problem can be interpreted differently. I assume the following:

- Productivity shock ε is drawn independently for every period, cohort and household. I.e. old parent can have different productivity from when he was young.
- Intra-vivos transfer i is chosen by parents when they are old, in the third period.
- Borrowing is not allowed.

1. Young parent:

$$V(a, i, h, \varepsilon, a_k, b) = \max_{n, c_y, e, e_k, s} u(c_y) + \beta \mathbb{E}_{a'|a,\varepsilon'} J(s, b, a_k, a', h'_o, h'_k, \varepsilon')$$
$$\text{s.t. } c_y + e + e_k + s = i + wh(1-n)\varepsilon$$
$$h'_o = a(nh)^{\gamma_1} e^{\gamma_2} + (1-\delta)h$$
$$h'_k = a_k h^{\gamma_1} e_k^{\gamma_2} + (1-\delta)h$$

Old parent:

$$J(s, b, a_k, a', h'_o, h'_k, \varepsilon') = \max_{c'_o, i', b'} u(c'_o) + \theta V(a_k, i', h'_k, \varepsilon', a', b')$$
$$\text{s.t. } c'_o + i' + b' = Rb + Rs + wh'_o\varepsilon'$$

2. *Plug in J into V, and use standard argument of Blackwell's sufficient condition (monotonicity and discounting) for V.*

3. *Plug J into V and apply Stokey-Lucas theorems.*

4. Plug in c_y, h'_o, h'_k, c_o from constraints.

   FOC:

$$\begin{aligned}
n: &\quad wh\varepsilon u'(c_y) = \beta \mathbb{E} J_5(t) \tfrac{dh'_o}{dn} \\
e: &\quad u'(c_y) = \beta \mathbb{E} J_5(t) \tfrac{dh'_o}{de} \\
e_k: &\quad u'(c_y) = \beta \mathbb{E} J_6(t) \tfrac{dh'_k}{de_k} \\
s: &\quad u'(c_y) = \beta \mathbb{E} J_1(t) \\
i': &\quad u'(c'_o) = \theta V_2(t+1) \\
b': &\quad u'(c'_o) = \theta V_6(t+1)
\end{aligned}$$

where

$$\frac{dh_o'}{dn} = \gamma_1 a n^{\gamma_1 - 1} h^{\gamma_1} e^{\gamma_2}$$
$$\frac{dh_o'}{dn} = \gamma_2 a (nh)^{\gamma_1} e^{\gamma_2 - 1}$$
$$\frac{dh_k'}{de_k} = \gamma_2 a_k (h)^{\gamma_1} e_k^{\gamma_2 - 1}$$

ENV:

$$\begin{aligned}
J_5(t) &= u'(c_o')w\varepsilon' \\
J_6(t) &= \theta V_3(t+1) \\
J_1(t) &= u'(c_o')R \\
V_2(t) &= u'(c_y) \\
V_6(t) &= \beta \mathbb{E} J_2(t) \\
J_2(t) &= Ru'(c_o') \\
V_3(t) &= u'(c_y)w(1-n)\varepsilon + \beta \mathbb{E}[J_5(t)\frac{dh_o'}{dh} + J_6(t)\frac{dh_k'}{dh}]
\end{aligned}$$

where

$$\frac{dh_o'}{dh} = \gamma_1 a n^{\gamma_1} h^{\gamma_1 - 1} e^{\gamma_2}$$
$$\frac{dh_k'}{dh} = \gamma_1 a_k (h)^{\gamma_1 - 1} e_k^{\gamma_2}$$

Euler equations:

$$\begin{aligned}
n:& \quad wh\varepsilon u'(c_y) = \beta \mathbb{E} u'(c_o') w \varepsilon' \tfrac{dh_o'}{dn} \\
e:& \quad u'(c_y) = \beta \mathbb{E} u'(c_o') w \varepsilon' \tfrac{dh_o'}{de} \\
e_k:& \quad u'(c_y) = \beta \mathbb{E} \tfrac{dh_k'}{de_k} \theta \left[u'(c_y') w (1-n')\varepsilon' + \beta u'(c_o'') w \varepsilon'' \tfrac{dh_o''}{dh'} + \beta \tfrac{dh_k''}{dh'} \theta V_3(t+2) \right] = ... \\
s:& \quad u'(c_y) = \beta \mathbb{E} R u'(c_o') \\
b':& \quad u'(c_o') = \theta \beta \mathbb{E} R u'(c_o'') \\
i':& \quad u'(c_o') = \theta u'(c_y')
\end{aligned}$$

Interpretations:

- n: Marginal benefit of working and hence earning more when young must be equal to marginal benefit of having higher human capital and hence earning more when old.
- e: Marginal cost of investing in own human capital and giving up consumption when young must be equal to marginal benefit of having higher human capital and hence earning more when old.

- e_k: Marginal cost of investing in child's human capital and giving up consumption when young must be equal to it's marginal benefit that comes in three ways: children earning more when they are young adults, children having higher human capital when they get old and hence earning more, and their children having higher human capital. The last term recursively unfolds into the infinite future, so optimal investment into children accounts for utilities of the whole dynasty.
- s: Marginal utility of consuming when young equals marginal utility of consuming more when old with added return on savings.
- b': Marginal utility of consumtion when old equals marginal utility of children next period when they become old, adjusted for how much parents care about children, θ.
- i': Marginal utility of old parents equals marginal utility of their children, adjusted by θ.

5. The difference, as discussed above, is that investment in children has far-reaching effect on all subsequent generations, whereas investment in own human capital only gives benefit in the next period.

6. There are two types of incompleteness in this environment. First, borrowing constraint: young parents can't borrow ($s \geq 0$, and old parents can't pass debt onto their heirs ($b \geq 0$). Second, there are no insurance markets, i.e. state-contingent claims. The first causes over-accumulation of physical and human capital due to a precautionary motive. Absence of contingent markets makes inter-temporal decisions (savings, investment in children, bequests) efficient ex ante (in expectation), but inefficient ex post. Under complete markets, savings will be lower. Investment into children will be such that their human capital only depends on their ability, but not on their parents' wealth. Allocation of resoures will be socially optimal.

7. Because the tax is proportional and the transfer is lump sum, such policy will result in redistribution from the rich (high productivity, high human capital, hence high labor earnings) to the poor (low productivity, low human capital). Redistribution effect will make the rich oppose such intervention.

   On the other hand, such fiscal policy provides a form of insurance and consumption smoothing over different realizations of productivity and ability of future generations. This insurance channel will be valuable for the rich.

8. In the standard Aiyagari-Bewley model agents are over-accumulating physical capital to protect themselves from a sequence of bad shocks in the future, because there is a potentially binding borrowing constraint. If technology exhibits diminishing product of capital, interest rate is inversely related to capital. Capital is above socially optimal level, so interest rate is below socially optimal (which is equal to the rate of time preference).

   In the model with human capital, this might not be true. As in Aiyagari-Bewley, market incompleteness makes agents over-accumulate capital, both physical and human. When technology uses both types of capital, interest rate depends negatively on physical capital, but positively on human capital. Depending on model parameters, either effect might dominate, so interest rates may be above or below the rate of time preference.

9. Notice that such redistribution was already available in the form of intra-vivos transfers. Now a certain amount T of such transfer becomes mandatory. It will not affect househods who's transfer was already big, $i > T$, they will merely reduce i by the amount of T. But other households - relatively poor parents of relatively rich children - who choose $i < T$, will be worse off, because they can't reduce their transfer i below zero.