Noah Williams

Department of Economics

University of Wisconsin

Problem Set 2
Due in Class on 2/24

1. This question considers an international model of endowment economies, with different trade arrangements. Assume that there are two countries, indexed by $i = a, b$. In each economy, there are two types of consumption goods, c_1 and c_2. An economy can be thought of as a representative agent with preferences over uncertain streams of the consumption goods given by:

$$E \sum_{t=0}^{\infty} \beta^t \frac{\left(c_{1t}^{\alpha} c_{2t}^{1-\alpha}\right)^{1-\gamma}}{1-\gamma} \qquad (1)$$

where $\beta \in (0,1)$, $\alpha \in (0,1)$, $\gamma > 0$. Preferences are the same across economies. In economy i the representative agent owns one tree that yields *both* types of consumption goods in each period, a random amount e_{1t}^i of good one and a constant amount 1 of good 2. Each period e_{1t}^a takes on values $e_h > 1/2$ with probability p or $1 - e_h$ with probability $1 - p$. Further suppose that the total world endowment of good 1 is constant at $e_{1t}^a + e_{1t}^b = 1$. Assume that good two is the numeraire, so that its price is always 1. Suppose that in addition to the goods market, there is a market for claims to trees, and a market for riskless one period bonds.

   (a) Suppose that the countries do not trade at all, in either goods or assets (a condition known as autarky). Define a competitive equilibrium in this environment.

   (b) Formulate a representative agent's problem and find the conditions for maximization. (Many of these conditions do not depend on the market structure.)

   (c) Under autarky, find expressions (or Euler equations) for the equilibrium prices in both countries of the spot price of good 1, the prices of claims to trees, and the equilibrium one period riskless interest rates (call these r_{At}^i for "autarky").

   (d) Now suppose that there is full trade in goods and assets in every period. Find new expressions for the riskless interest rates (call these r_{ft}^i, for "free trade"). Do the interest rates vary across countries?

   (e) Now suppose that the countries can trade in goods, but not assets every period. What does this imply for the prices of consumption goods across countries? What are the relevant market clearing conditions now?

   (f) Continue with trade in goods only. Since there is no international borrowing and lending, in equilibrium the value of a country's consumption (at international market prices) must equal the value of its endowment (again at international market prices) every period. Find new expressions for the equilibrium spot price of good 1, and one period riskless interest rates across countries (call these r_{gt}^i, for "goods trade"). Do the interest rates vary across countries?

(g) Suppose that $\alpha = 1/2, \gamma = 3$ and $e_h = 3/4$. For country a, analyze (at least qualitatively) the level and the variability of the interest rates r^a_{At}, r^a_{gt}, and r^a_{ft}. Interpret your results.

2. Consider an optimal growth model in which households value government spending as well as private consumption. That is, suppose that households have the preferences:

$$\sum_{t=0}^{\infty} \beta^t \frac{(c_t G_t^\eta)^{1-\gamma}}{1 - \gamma}$$

where c_t is private consumption, G_t is government spending, and η is a substitution parameter that may be positive or negative. Assume $\beta \in (0, 1)$ and $\gamma > 1$. Government spending follows an exogenous path which is known with certainty at date 0. The government spending is financed by lump sum taxes. A benevolent social planner seeks to maximize agents' utility subject to the feasibility constraint:

$$k_{t+1} = (1 - \delta)k_t + f(k_t) - c_t - G_t$$

with k_0 given. The social planner cannot alter the path of government spending. The production function $f(k)$ is strictly concave and satisfies:

$$\lim_{k \to 0} f'(k) = +\infty, \quad \lim_{k \to +\infty} f'(k) = 0.$$

(a) Formulate the Bellman equation for the social planner's problem and derive the conditions for maximization. Find the difference equations governing the evolution of consumption and capital along an optimal path.

(b) When government spending grows at a constant rate g, is there a steady state in this economy? Is it unique?

(c) What happens to consumption and capital if there is a once and for all unexpected increase in g, the growth rate of government spending? How does this depend on the agents' preferences for government spending? Consider both the transitional dynamics (qualitatively) and long-run effects (analytically), and interpret your results.

3. Consider a recursive version of the neoclassical model with distorting taxes. An infinitely-lived representative household owns a stock of capital k which it rents to firms. The household's capital stock depreciates are rate δ, and denote aggregate capital by K. Households do not value leisure and are endowed with one unit of time each period with which they can supply labor N to firms. They have standard time additive expected utility preferences with discount factor β and period utility $u(c)$. Firms produce output according to the production function $zF(K, N)$ where z is the stochastic level of technology which is Markov with transition function $P(z', z)$. There is also a government which levies a proportional tax τ on households' capital income and labor income. The tax rate is the same for both types of income and constant over time. The government uses the proceeds of the taxes to provide households with a lump sum transfer payment T (which may vary over time), and suppose that in equilibrium the government must balance its budget every period.

(a) Define a recursive competitive equilibrium for this economy. Be specific about all of the objects in the equilibrium and the conditions they must satisfy.

(b) Characterize the equilibrium by finding a functional (Euler) equation which the household's optimal capital accumulation policy must satisfy (i.e. k' as a function of k, K, z).

(c) Impose the equilibrium conditions and find a functional equation which the aggregate capital accumulation policy must satisfy.

(d) Show that when $\delta = 1$, the recursive competitive equilibrium allocation coincides with the solution of a social planner's problem, but with a different discount factor. Interpret your result.

4. Consider the neoclassical growth model with linear taxation as in class, but instead of there being taxation on capital and labor income, there are linear taxes τ_t^c on consumption. That is, the household has preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - N_t)$$

and the household budget constraint (with date-0 trading) becomes:

$$\sum_{t=0}^{\infty} q_t[(1 + \tau_t^c)c_t + k_{t+1} - (1 - \delta)k_t] = \sum_{t=0}^{\infty} q_t[w_t N_t + r_t k_t].$$

As always, firms operate in a competitive market and produce via a constant returns to scale production function $F(k, N)$, and the aggregate feasibility condition is:

$$c_t + k_{t+1} - (1 - \delta)k_t + G_t = F(k_t, N_t).$$

(a) Show a consumption tax scheme is equivalent, in terms of having the same effect on the household's optimality conditions, to a particular choice of capital and labor income taxes.

(b) **For this part only** suppose that leisure is not valued, so labor supply is identically equal to one. Suppose that consumption taxes are increasing over time at a constant rate, so $1 + \tau_{t+1}^c = (1 + g)(1 + \tau_t^c)$, and this finances a constant level of government spending $G_t = G$. (Assume that (g, G) are always set so that the government's budget balances.) What happens to capital and consumption when both (g, G) are increased? Consider both the steady state and the transitional dynamics.

(c) Reconsider the case of valued leisure as in part (a). Suppose the government optimally chooses the consumption tax to maximize the agent's utility. Characterize the optimal consumption tax (its level and how it varies over time) in a steady state of such a Ramsey equilibrium.